Instructor: Professor, Ali Ghaffari	*Nonlinear Control and Adaptive Systems* **Project No. 01** **Due Date: 2019/March/06**	School of Mechanical Engineering Dynamics and Control 2018-2019 Teaching Assistant: Hamid Rahmanei

مهم: توضیحات پروژه اوّل **مهم**

بایدها و نبایدهای پروژه:

۱- فایل‌های Word و Pdf و کدهای MATLAB مربوط به پروژه خود را به صورت طبقه‌بندی‌شده و مرتب در یک پوشه قرار دهید و آن را زیپ کنید و با فرمت زیر به اینجانب تحویل بدهید یا ایمیل کنید.

مثلاً: *hamidrahmanei_8704463_Nonlinear_ project_01.rar*

۲- لطفاً مقاله پیوست را به دقّت مطالعه کنید و متن کامل آن را به فارسی روان، ترجمه کنید.

۳- فرمت گزارش کار ترجمه حتماً مشابه مقاله اصلی و در ۲ ستون تهیّه شود.

۴- علاوه بر ترجمه متن کامل مقاله، مثال‌های ارائه شده در آخر مقاله را نیز حل کنید و توضیح بدهید. هم‌چنین نمودار مسیرهای حرکت سیستم در هر دو مثال را به کمک یک کد مناسب در نرم‌افزار MATLAB ترسیم کنید و نتایج را با حل خود مقایسه کنید.

۵- هر داده‌ای که در صورت مسئله‌ها ذکر نشده و در شبیه‌سازی‌ها و حل مسئله مورد نیاز است, به دلخواه انتخاب کنید و در گزارش خود توضیح دهید.

۶- اگر در انجام هر قسمت از مراحل پروژه به مشکل برخوردید و یا نیاز به راهنمایی داشتید, می‌توانید با هماهنگی قبلی از طریق ایمیل *hamid.rahmanei20@gmail.com*, به اینجانب در آزمایشگاه کنترل و روباتیک (طبقه ۱) مراجعه کنید.

۷- لازم به تذکر است که همه گزارش‌ها باید تایپ‌شده باشند و دستنویس قابل قبول نیست!

۸- حتماً انجام این پروژه را به تاخیر نیندازید! حجم تمرین‌های این درس و دروس دیگر بسیار زیاد بوده و از طرفی تحویل پس از موعد مقرّر قابل قبول نیست. لطفاً زمانبندی مناسبی برای پروژه شماره ۱ انجام بدهید.

مقاله اصلی در صفحه بعدی پیوست شده است.

(این مقاله در سال ۲۰۰۹، در ژورنال Nonlinear Dynamics (**IF=3.46; Q1**) که معتبرترین ژورنال در زمینه سیستم‌های دینامیکی غیرخطّی است، چاپ شده است)

Nonlinear Dyn (2009) 56: 269–275
DOI 10.1007/s11071-008-9398-3

ORIGINAL PAPER

The stability of limit cycles in nonlinear systems

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Received: 12 March 2007 / Accepted: 10 July 2008 / Published online: 23 September 2008

Abstract In this paper, a necessary condition is first presented for the existence of limit cycles in nonlinear systems, then four theorems are presented for the stability, instability, and semistabilities of limit cycles in second order nonlinear systems. Necessary and sufficient conditions are given in terms of the signs of first and second derivatives of a continuously differentiable positive function at the vicinity of the limit cycle. Two examples considering nonlinear systems with familiar limit cycles are presented to illustrate the theorems.

Keywords Limit cycle · Existence · Stability · Nonlinear systems · First and second derivatives

1 Introduction

The existence of limit cycles and their stability analysis in nonlinear systems have always been of interest for mathematicians and dynamic system engineers. Since Van der Pol studied a second order nonlinear differential equation and proved that the system had a limit cycle, his results have been extended by many researchers. Although there have been extensive researches in this field, but the nonlinear characteristics of these systems have made the distinguished results very limited. There are some limited numbers of theorems regarding the existence of limit cycles in nonlinear systems. According to the Poincaré theorem, for second-order autonomous differential equations in the form of $dx(t)/dt = f(x)$, if a limit cycle exists, then $N = S + 1$, where N is the number of nodes, centers, and foci enclosed by a limit cycle, and S is the number of enclosed saddle points [1]. Another theorem presented by Poincaré and Bendixson is concerned with the asymptotic properties of the trajectories of second order systems [2]. Bendixson also presented a theorem as a sufficient condition for nonexistence of limit cycles [3]. According to this theorem, for a second order nonlinear system in the form of $dx(t)/dt = f(x)$, where $f(x) = [f_1(x)\ \ f_2(x)]^{\mathrm{T}}$, no limit cycle exists in a region where $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ does not vanish and does not change sign [3]. Dragilev proposed some theorems regarding the existence of a stable (unstable) limit cycle for the Liénard equation [4, 5]. In [6], a survey regarding the researches done since 1882 with the main subject of limit cycles is presented. It contains more than 300 published papers or books.

In this paper, we first present a necessary condition for the existence of limit cycles, then we present four theorems for the stability, instability, and semista-

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bilities of limit cycles in second order nonlinear systems. Necessary and sufficient conditions are proven and two examples are given to illustrate them. Two regions are defined inside and outside of a limit cycle where the stability of the limit cycle depends on the signs of the first and second derivatives of a positive function in these regions. Since the terms "inside and outside of a closed curve" is only defined for second-order systems [7], therefore, the stability theorems are limited to this class of nonlinear systems.

2 A necessary condition for existence of limit cycles

Consider an autonomous unforced nonlinear system:

$$dx(t)/dt = f(x) \quad f: D \to R^n \tag{1}$$

where D is an open and connected subset of R^n and f is a locally Lipschitz map from D into R^n. Let the point $x = x_e = 0$ be an equilibrium point of (1) and assume that L is a limit cycle of (1) and $g(x) = 0$ be the equation of the limit cycle L.

If $g(x)$ is a continuously differentiable function (on the limit cycle L and at its vicinity), then there exists a continuously differentiable scalar function $V(x) \geq 0$, $V: D \to R$ at the vicinity of the limit cycle such that on the limit cycle we have:

$$dV(x)/dt = d^2V(x)/dt^2 = 0 \tag{2}$$

for all $x \in L$.

2.1 Proof

A limit cycle is defined as an isolated closed curve, where any trajectory started on this curve will stay on it forever [8]. In other words, trajectories on the limit cycle never leave it. Since $g(x) = 0$ represents the equation of the limit cycle and $g(x)$ is assumed to be a continuously differentiable function, therefore, according to the invariant set theorem [8], we have

$$dg(x)/dt = 0 \tag{3}$$

for all $x \in L$.

Note that on the limit cycle we also have

$$d^2g(x)/dt^2 = 0 \tag{4}$$

We prove this by contradiction. Let $d^2g(x)/dt^2 \neq 0$ in the whole region or some part of the limit cycle. Then it follows that $dg(x)/dt$ changes from zero in this region. It contradicts the fact that $dg(x)/dt$ is always zero on the limit cycle as stated in (3). Let's define the function $V(x) \geq 0$ on the limit cycle and at its vicinity as:

$$V(x) = \left[g(x) - g(0)\right]^n \tag{5}$$

where n is an arbitrary positive even integer, and $g(0)$ is the value of $g(x)$ at the equilibrium point $x = 0$. With this definition, clearly $V(x)$ is a positive scalar continuously differentiable function. Now we calculate the first and second time derivatives of $V(x)$ to show that they are zero on the limit cycle L.

$$dV(x)/dt = n\left[g(x) - g(0)\right]^{n-1} \cdot dg(x)/dt \tag{6}$$

$$\begin{aligned} d^2V(x)/dt^2 = {} & n(n-1)\left[g(x) - g(0)\right]^{n-2} \\ & \cdot \left(dg(x)/dt\right)^2 \\ & + n\left[g(x) - g(0)\right]^{n-1} \\ & \cdot d^2g(x)/dt^2 \end{aligned} \tag{7}$$

The first derivative $dV(x)/dt$ in (6) is zero on the limit cycle since $dg(x)/dt$ is zero. Also, the second derivative $d^2V(x)/dt^2$ in (7) is zero since both $dg(x)/dt$ in the first term of the right-hand side and $d^2g(x)/dt^2$ in the second term are zero according to (3) and (4).

2.2 Some remarks about $V(x)$

1. Note that the function $V(x)$ in this theorem is not unique. Any continuously differentiable positive function with its first and second derivatives equal to zero only on the limit cycle can be considered as $V(x)$.
2. It is also important to note that if $V(x) \leq 0$, it is still possible to draw the conclusion of $dV(x)/dt = d^2V(x)/dt^2 = 0$ for all $x \in L$. However, we consider $V(x) \geq 0$, since we need this condition when we are analyzing the stability of limit cycles in the next section.
3. If $[g(x) - g(0)] \geq 0$, then we may set $n = 1$ in (5) and we have

$$V(x) = \left[g(x) - g(0)\right] \tag{8}$$

4. Note that the function $V(x)$ should be selected to be constant only on the limit cycle. At the vicinity of the limit cycle, $V(x)$ is a function of the state variables of the system, and is not constant.

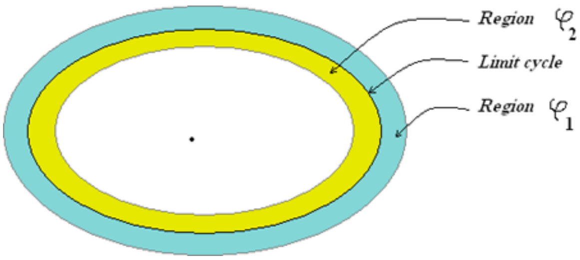

Fig. 1 Two Regions ϕ_1 and ϕ_2 (outside and inside vicinities) of the limit cycle L in the phase plane

3 Stability analyses of limit cycles

Here, we present the necessary and sufficient conditions for the stability, instability, and semi-stabilities of limit cycles. However, we first prove the sufficiency of the theorems and then we prove that these theorems are also necessary conditions. In order to present these theorems, we need to investigate and evaluate the signs of both $dV(x)/dt$ and $d^2V(x)/dt^2$ at the vicinity of a limit cycle L. However, since in this part we deal with the concept of inside and outside of a limit cycle, we limit ourselves to second order nonlinear systems and we assume that in (1), we have $n = 2$. Let us define two regions at the outside and inside vicinities of the limit cycle L. Figure 1 shows the limit cycle L and the regions ϕ_1 and ϕ_2 as the outside and inside vicinities of the limit cycle, respectively.

The closed and bounded set ϕ_1, as the region outside the limit cycle, is defined such that any point $X\phi_1$ in ϕ_1 is a member of the neighborhood $Br_1(X_L)$ where X_L is an arbitrary point on the limit cycle and we have $\|X\phi_1 - X_L\| < r_1$ and $\|X\phi_1\| > \|X_L\|$. The quantity r_1 is the radius of a circle indicating the neighborhood radius. Note that with this definition any point in ϕ_1 is a member of $Br_1(X_L)$ but any point at the neighborhood of X_L does not necessarily belong to ϕ_1. A similar procedure is used to define ϕ_2 as the inside vicinity of the limit cycle. Based on the above expressions, we now present the stability theorems for limit cycles.

4 Theorem 1: stability of limit cycles

The limit cycle L is stable if and only if the signs of $dV(x)/dt$ and $d^2V(x)/dt^2$ at the vicinities ϕ_1 and ϕ_2 of the limit cycle L are:

$$dV(x)/dt < 0, \qquad d^2V(x)/dt^2 > 0 \tag{9}$$

outside limit cycle and at its vicinity in region ϕ_1 (except on the limit cycle, where both derivatives are zero)

$$dV(x)/dt > 0, \qquad d^2V(x)/dt^2 < 0 \tag{10}$$

inside limit cycle and at its vicinity in region ϕ_2 (except on the limit cycle, where both derivatives are zero).

Here $V(x)$ is the positive continuously differentiable function as defined in Sect. 2.

4.1 Proof

In the first step, we prove the sufficiency of the theorem. According to Sect. 2, there exists a scalar positive continuously differentiable function $V(x)$ at the vicinity of the limit cycle L such that on the limit cycle we have $dV(x)/dt = d^2V(x)/dt^2 = 0$. To prove the stability of the limit cycle, we show that if inequalities (9) and (10) hold, all the trajectories in the vicinity of the limit cycle both in regions ϕ_1 and ϕ_2, will converge to the limit cycle.

1. Region ϕ_1 outside limit cycle L:

Let us first consider trajectories in region ϕ_1 (outside limit cycle and at its vicinity). In this region, according to (9), we have $dV(x)/dt < 0$ and $d^2V(x)/dt^2 > 0$. Thus, $V(x)$ is positive decreasing and $dV(x)/dt$ is a negative increasing function along trajectory. Noting that on the limit cycle we have $dV(x)/dt = 0$, therefore, trajectories in region ϕ_1 will converge to the limit cycle, as $t \to \infty$. Also note that according to Barbalat's lemma, $V(x)$ is uniformly continuous since its derivative $dV(x)/dt$ is bounded [8].

2. Region ϕ_2 inside limit cycle L:

In this region, according to (10), we have $dV(x)/dt > 0$ and $d^2V(x)/dt^2 < 0$. Therefore, $V(x)$ is positive increasing and $dV(x)/dt$ is a positive decreasing function along any trajectory. Noting that on the limit cycle we have $dV(x)/dt = 0$, thus trajectories in region ϕ_2 will converge to the limit cycle as $t \to \infty$.

An important point to be determined is the signs of these derivatives at the vicinity of the equilibrium point. Note that according to the Poincaré theorem [1], inside the limit cycle there exists an equilibrium point. (In our case, the point $x = 0$ is assumed to be the equilibrium point of the nonlinear system (1).)

For a stable limit cycle, the equilibrium point is unstable and trajectories at its vicinity diverge from it and converge to the limit cycle. Therefore, inside limit cycle, both at the vicinity of the equilibrium point and in region ϕ_2, we have $dV(x)/dt > 0$. However, $dV(x)/dt$ is zero both at the equilibrium point and on the limit cycle. Nothing that $dV(x)/dt$ is a continuously differentiable function, thus there must be a point inside limit cycle where $dV(x)/dt$ is maximum. Clearly, at this point, $d^2V(x)/dt^2 = 0$ and at this point the sign of $d^2V(x)/dt^2$ changes. In fact, this point is on the lower boundary of ϕ_2. This completes the sufficiency of the stability theorem of the limit cycles. We proceed with the instability and semistability theorems and then we prove the necessity of each theorem.

5 Theorem 2: instability of limit cycles

The limit cycle L is unstable if and only if the signs of $dV(x)/dt$ and $d^2V(x)/dt^2$ at the vicinities ϕ_1 and ϕ_2 of the limit cycle L are

$$dV(x)/dt > 0, \qquad d^2V(x)/dt^2 > 0 \tag{11}$$

outside limit cycle and at its vicinity in region ϕ_1 (except on the limit cycle, where both derivatives are zero)

$$dV(x)/dt < 0, \qquad d^2V(x)/dt^2 < 0 \tag{12}$$

inside limit cycle and at its vicinity in region ϕ_2 (except on the limit cycle, where both derivatives are zero).

Here $V(x)$ is the positive continuously differentiable function as defined in Sect. 2.

5.1 Proof

Outside limit cycle and at its vicinity in region ϕ_1, as given by (11), we have $dV(x)/dt > 0$ and $d^2V(x)/dt^2 > 0$. Therefore, $dV(x)/dt$ is a positive increasing function and it does not converge to zero. Noting that on the limit cycle, we have $dV(x)/dt = 0$, therefore, trajectories in ϕ_1 do not converge to the limit cycle. It means that either the limit cycle is unstable or semistable.

Inside limit cycle in region ϕ_2, as given by (12), we have $dV(x)/dt < 0$ and $d^2V(x)/dt^2 < 0$. It means that as time proceeds, the negative function $dV(x)/dt$ becomes smaller and smaller. Therefore, it does not converge to zero. So, trajectories in ϕ_2 do not converge to the limit cycle. This completes the sufficiency of the instability theorem. Similar arguments may be given for the signs of $dV(x)/dt$ and $d^2V(x)/dt^2$ inside an unstable limit cycle and at the vicinity of the equilibrium point. Since the equilibrium is stable in the sense of Lyapunov, therefore, $dV(x)/dt$ is negative inside limit cycle. However, it is zero both at the equilibrium point and on the limit cycle. Noting that $dV(x)/dt$ is a continuous function it has a minimum inside limit cycle. The point that $dV(x)/dt$ is minimum, is located on the lower boundary of ϕ_2. At this point, $d^2V(x)/dt^2 = 0$ and its sign changes.

Before we proceed with our discussion, it should be noticed that depending on the motion pattern of trajectories in the vicinity of a limit cycle, two types of semistable limit cycles maybe defined. In the next section, we will define them.

6 Definitions of semi-stable limit cycle

By definition, a limit cycle is semistable if some of the trajectories in the vicinity converge to it and the others diverge from it as $t \to \infty$ [8]. Based on this definition, we may have two different trajectory patterns for second order systems at the vicinity of a limit cycle, where both of them indicate semistable limit cycles.

Definition 1 A limit cycle is defined to be semi-stable type-1, if the trajectories outside limit cycle converge to it and those inside limit cycle diverge from it as $t \to \infty$.

Definition 2 A limit cycle is defined to be semi-stable, type-2, if the trajectories outside limit cycle diverge from it and those inside limit cycle converge to it as $t \to \infty$.

Based on these definitions, the two semistability theorems for limit cycles are presented here.

7 Theorem 3: semistability type-1 of limit cycles

The limit cycle L is semistable type-1 if and only if the signs of $dV(x)/dt$ and $d^2V(x)/dt^2$ at the vicinities ϕ_1 and ϕ_2 of the limit cycle L are

$$dV(x)/dt < 0, \qquad d^2V(x)/dt^2 > 0 \tag{13}$$

outside limit cycle and at its vicinity in region ϕ_1 (except on the limit cycle, where both derivatives are zero)

$$dV(x)/dt < 0, \qquad d^2V(x)/dt^2 < 0 \tag{14}$$

inside limit cycle and at its vicinity in region ϕ_2 (except on the limit cycle, where both derivatives are zero)

7.1 Proof

The inequality (13) is the same as (9). Thus, the trajectories outside limit cycle have the same pattern as stable limit cycle and they converge to the limit cycle. Similarly, the inequality (14) is the same as (12). Therefore, the trajectories inside the limit cycle have the same pattern as unstable limit cycles and they diverge from it. Therefore, the limit cycle is semistable type-1 as stated by Definition 1. It is important to note that according to semistability Theorem 1, $dV(x)/dt$ is always negative at the vicinity of the limit cycle and it is zero on the limit cycle. It means that the equation $dV(x)/dt = 0$ has repeated roots on the limit cycle.

8 Theorem 4: semistability type-2 of limit cycles

The limit cycle L is semistable type-2 if and only if the signs of $dV(x)/dt$ and $d^2V(x)/dt^2$ at the vicinities ϕ_1 and ϕ_2 of the limit cycle L are

$$dV(x)/dt > 0, \qquad d^2V(x)/dt^2 > 0 \tag{15}$$

outside limit cycle and at its vicinity in region ϕ_1 (except on the limit cycle, where both derivatives are zero)

$$dV(x)/dt > 0, \qquad d^2V(x)/dt^2 < 0 \tag{16}$$

inside limit cycle and at its vicinity in region ϕ_2 (except on the limit cycle, where both derivatives are zero).

8.1 Proof

The proof is straightforward. The inequality (15) is the same as (11). Thus, trajectories outside limit cycle have the same pattern as unstable limit cycles and they diverge from it as $t \to \infty$. Similarly, the inequality (16) is the same as (10). Thus, trajectories inside the limit cycle have the same pattern as stable limit cycles and they converge to the limit cycle as $t \to \infty$. Therefore, the limit cycle is semistable, type-2 as stated by Definition 2. Note that in this case the function $dV(x)/dt$ is positive both inside and outside limit cycle, but is zero on the limit cycle. Therefore, the function $dV(x)/dt = 0$ has repeated roots on the limit cycle.

9 Proof of necessities

Up to this point, we have proved the sufficiency of the theorems. In order to prove their necessities, we investigate the trajectory patterns at the vicinity of a hypothetical limit cycle and we determine the signs of $dV(x)/dt$ and $d^2V(x)/dt^2$ to match those trajectory patterns. Since the scalar function $V(x)$ is a positive continuously differentiable function and also noting that on the limit cycle both $dV(x)/dt$ and $d^2V(x)/dt^2$ are zero, the trajectory patterns outside and inside a limit cycle and the signs of $dV(x)/dt$ and $d^2V(x)/dt^2$ for these trajectories are determined as follows:

9.1 Outside limit cycle (in region ϕ_1)

In region ϕ_1, two cases may happen:

1. Trajectories are diverging from the limit cycle then both $dV(x)/dt$ and $d^2V(x)/dt^2$ are positive.
2. Trajectories are converging to the limit cycle as $t \to \infty$, then $dV(x)/dt$ is negative and $d^2V(x)/dt^2$ is positive.

9.2 Inside limit cycle (in region ϕ_2)

In region ϕ_2, two cases may happen:

1. Trajectories are converging to the limit cycle as $t \to \infty$, then $dV(x)/dt$ is positive and $d^2V(x)/dt^2$ is negative.
2. Trajectories are diverging from the limit cycle then both $dV(x)/dt$ and $d^2V(x)/dt^2$ are negative.

Table 1 illustrates the acceptable signs for $dV(x)/dt$ and $d^2V(x)/dt^2$ both at the outside and inside vicinities of a limit cycle. Considering all possible 16 cases for the signs of $dV(x)/dt$ and $d^2V(x)/dt^2$ at the vicinities of a hypothetical limit cycle, only four

Table 1 Acceptable signs for $dV(x)/dt$ and $d^2V(x)/dt^2$ at the vicinity of a limit cycle in regions ϕ_1 and ϕ_2

	Regions ϕ_2	Regions ϕ_1
$dV(x)/dt$	+	+
$d^2V(x)/dt^2$	−	+
$dV(x)/dt$	−	−
$d^2V(x)/dt^2$	−	+

cases are acceptable since the signs of $dV(x)/dt$ and $d^2V(x)/dt^2$ are among those in Table 1.

Signs of $dV(x)/dt$ and $d^2V(x)/dt^2$ in Table 1, comparing with the 16 possible cases for $dV(x)/dt$ and $d^2V(x)/dt^2$ in regions ϕ_1 and ϕ_2 indicate that the signs of $dV(x)/dt$ and $d^2V(x)/dt^2$, are the only required information for the stability, instability, and semistabilities of a limit cycle. This completes the necessities of the theorems.

10 Conclusion

In this paper, a necessary condition is first presented for the existence of limit cycles in nonlinear systems. Then four theorems are presented and proved as necessary and sufficient conditions for the stability analysis of limit cycles. The stability analysis depends on the signs of first and second time derivatives of a continuously differentiable positive function at the vicinity of the limit cycles. Two examples of nonlinear systems with familiar limit cycles illustrate the theorems and their applications.

Appendix

The following examples represent four systems which have limit cycles [8, 9].

Example 1

$$dx_1/dt = x_2 + k_1x_1\left(x_1^2 + x_2^2 - \beta^2\right)$$
$$dx_2/dt = -x_1 + k_2x_2\left(x_1^2 + x_2^2 - \beta^2\right)$$

such that $\beta \neq 0$ and $k_1 = k_2 = +1$ or $k_1 = k_2 = -1$ the point $x_1 = 0, x_2 = 0$ is the equilibrium point of the system. Let's define the positive continuously differentiable function $V(x)$ as:

$$V(x) = x_1^2 + x_2^2$$

then we have:

$$dV(x)/dt = 2\left(x_1^2 + x_2^2 - \beta^2\right)\left(k_1x_1^2 + k_2x_2^2\right)$$
$$d^2V(x)/dt^2 = 4\left(x_1^2 + x_2^2 - \beta^2\right)\left[2k_1x_1^4 + 2k_2x_2^4 + (k_1 + k_2)^2x_1^2x_2^2 - \beta^2\left(k_1^2x_1^2 + k_2^2x_2^2\right)\right]$$

According to the stability theorem (Sect. 4) and instability theorem (Sect. 5) and (9), (10), (11), and (12), we can determine the stability or instability of the limit $x_1^2 + x_2^2 - \beta^2$. We have

For $k_1 = k_2 = -1$, the limit cycle $x_1^2 + x_2^2 - \beta^2$ is stable.

For $k_1 = k_2 = +1$, the limit cycle $x_1^2 + x_2^2 - \beta^2$ is unstable.

Example 2

$$dx_1/dt = x_2 + k_1x_1\left(x_1^2 + x_2^2 - \beta^2\right)^2$$
$$dx_2/dt = -x_1 + k_2x_2\left(x_1^2 + x_2^2 - \beta^2\right)^2$$

such that $\beta \neq 0$ and $k_1 = k_2 = +1$ or $k_1 = k_2 = -1$ the point $x_1 = 0, x_2 = 0$ is the equilibrium point of the system. Let's define the positive continuously differentiable function $V(x)$ as

$$V(x) = x_1^2 + x_2^2$$

then we have

$$dV(x)/dt = 2\left(x_1^2 + x_2^2 - \beta^2\right)^2\left(k_1x_1^2 + k_2x_2^2\right)$$
$$d^2V(x)/dt^2 = 8\left(x_1^2 + x_2^2 - \beta^2\right)^3\left(k_1x_1^2 + k_2x_2^2\right)^2 + 4(x_1^2 + x_2^2 - \beta^2)^4\left(k_1^2x_1^2 + k_2^2x_2^2\right)$$

According to the semistability theorems (Sects. 7 and 8) and (13), (14), (15), and (16), we can determine the semistabilities of the limit $x_1^2 + x_2^2 - \beta^2$. We have

For $k_1 = k_2 = -1$, the limit cycle $x_1^2 + x_2^2 - \beta^2$ is semistable type-1.

For $k_1 = k_2 = +1$, the limit cycle $x_1^2 + x_2^2 - \beta^2$ is semistable type-2.

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